A duality result between the minimal surface equation and the maximal surface equation

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ABSTRACT

In this note we show how classical Bernstein's theorem on minimal surfaces in the Euclidean space can be seen as a consequence of Calabi-Bernstein's theorem on maximal surfaces in the Lorentz-Minkowski space (and viceversa). This follows from a simple but nice duality between solutions to their corresponding differential equations.

Key words: Minimal surface equation, Maximal surface equation, Bernstein's theorem, Calabi-Bernstein's theorem.

1. INTRODUCTION

A minimal surface in Euclidean space $\mathbb{R}^3$ is a surface with zero mean curvature. Bernstein (1915-1917) proved that the planes are the only minimal entire graphs in $\mathbb{R}^3$.

THEOREM 1. (BERNSTEIN'S THEOREM). *The only entire solutions to the minimal surface equation*

$$\text{Minimal}[u] = \text{Div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

are affine functions.

On the other hand, a maximal surface in the Lorentz-Minkowski space $\mathbb{L}^3$ is a spacelike surface with zero mean curvature. Here by *spacelike* we mean that the induced metric from the Lorentzian

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metric in $\mathbb{L}^3$ is a Riemannian metric on the surface. Calabi (1970) obtained the corresponding version of Bernstein's theorem for the case of maximal surfaces.

THEOREM 2. (CALABI-BERNSTEIN'S THEOREM). *The only entire solutions to the maximal surface equation*

$$\text{Maximal}[u] = \text{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad |Du|^2 < 1,$$

are affine functions.

Here the condition $|Du|^2 < 1$ means precisely that the graph defined by u is spacelike.

In this note we show how classical Bernstein's theorem on minimal surfaces in the Euclidean space $\mathbb{R}^3$ can be seen as a consequence of Calabi-Bernstein's theorem on maximal surfaces in the Lorentz-Minkowski space $\mathbb{L}^3$ (and viceversa). This follows from the following duality between solutions to their corresponding differential equations.

THEOREM 3. *Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected domain. There exists a non-affine C^2 solution to the minimal surface equation on Ω*

$$\text{Minimal}[u] = \text{Div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

if and only if there exists a non-affine C^2 solution to the maximal surface equation on Ω

$$\text{Maximal}[w] = \text{Div}\left(\frac{Dw}{\sqrt{1-|Dw|^2}}\right) = 0, \qquad |Dw|^2 < 1.$$

2. PROOF OF THEOREM 3

PROOF. Assume that u is a non-affine solution of $\text{Minimal}[u] = 0$ on the domain Ω. Recall that for a vector field X on $\mathbb{R}^2$ it holds that

$$(\text{Div}X)dx \wedge dy = d\omega_{JX},$$

where J denotes the positive $\pi/2$-rotation in the plane and ω_{JX} denotes the 1-form on $\mathbb{R}^2$ which is metrically equivalent to the field JX, that is, ω_{JX} satisfies

$$\omega_{JX}(Y) = \langle JX, Y\rangle$$

for every vector field Y on $\mathbb{R}^2$. Then $\text{Minimal}[u] = 0$ is equivalent to the fact that ω_{JU} is closed on Ω, where U is the field on Ω given by

$$U = \frac{Du}{\sqrt{1+|Du|^2}}.$$

Then since the domain Ω is simply connected, we can write

$$J\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = Dw \tag{1}$$

for a certain C^2 function w on Ω. Since J is an isometry, there follows

$$|Dw|^2 = \frac{|Du|^2}{1+|Du|^2} < 1, \tag{2}$$

and also

$$1+|Du|^2 = \frac{1}{1-|Dw|^2}. \tag{3}$$

From (2), we see that w satisfies the spacelike condition. Besides, using that $J^2 = -\mathrm{id}$, we obtain from (1) and (3) that

$$J\left(\frac{Dw}{\sqrt{1-|Dw|^2}}\right) = \sqrt{1+|Du|^2}J(Dw) = D(-u),$$

and so Maximal$[w] = 0$ follows on Ω.

If w were affine, then Dw is a constant vector, $|Dw|^2 \equiv$constant, and then it follows from (3) that $|Du|^2$ is a constant also. It then follows from (1) that Du is a constant vector, contradicting the assumption that u is non-affine.

A very similar argument, starting with a non-affine solution of Maximal$[w] = 0$ on Ω with $|Dw|^2 < 1$, produces a non-affine solution of Minimal$[u] = 0$ on Ω. □

In particular, when Ω is the whole plane $\mathbb{R}^2$ we obtain the following.

COROLLARY 4. *There exists an entire, non-affine C^2 solution to the minimal surface equation*

$$\text{Minimal}[u] = \text{Div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

on $\mathbb{R}^2$ if and only if there exists an entire, non-affine C^2 solution to the maximal surface equation

$$\text{Maximal}[w] = \text{Div}\left(\frac{Dw}{\sqrt{1-|Dw|^2}}\right) = 0, \qquad |Dw|^2 < 1$$

on $\mathbb{R}2$.

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RESUMO

Nesta nota, mostramos como o clássico teorema de Bernstein sobre as superfícies mínimas no espaço Euclideano pode ser visto como uma consequência do teorema de Calabi-Bernstein sobre as superfícies máximas no espaço de Lorentz-Minkowski (e vice-versa). Isto decorre de uma simples, mas elegante, dualidade entre soluções a suas correspondentes equações diferenciais.

Palavras-chave: Equações de superfícies mínimas, Equações de superfícies máximas, teorema de Bernstein, teorema de Calabi-Bernstein.

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