$O(p+1) \times O(p+1)$-Invariant Hypersurfaces with Zero Scalar Curvature in Euclidean Space

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ABSTRACT

We use equivariant geometry methods to study and classify zero scalar curvature $O(p+1) \times O(p+1)$-invariant hypersurfaces in R^{2p+2} with $p > 1$.

Key words: equivariant geometry, scalar curvature.

1. INTRODUCTION

The methods of equivariant geometry have been applied successfully by many authors to obtain and classify explicit examples of hypersurfaces, with a given condition on the r-th mean curvature, that are invariant by the action of an isometry group (see, for instance, Hsiang *et al.* 1983, Hsiang 1982, do Carmo & Dajczer 1983, Bombieri *et al.* 1969, Alencar 1993).

O. Palmas (Palmas 1999), resuming a work started initially by T. Okayasu (Okayasu 1989) and using ideas contained in Alencar, 1993, published a work in which he approaches the hypersurfaces with zero scalar curvature in R^{2p+2}, invariant by the action of the group $O(p+1) \times O(p+1)$. In his article, Palmas studied only the case $p = 1$.

The objective of this work is to announce and give an sketch of proof of a classification theorem for the case $p > 1$. The *orbit space* of the action is the set $\Omega = \left\{(x, y) \in R^2; x \geq 0, y \geq 0\right\}$ and the invariant hypersurfaces are generated by curves $\gamma(t) = (x(t), y(t))$, the so called *profile curves*, that satisfy the following diferential equation

$$\begin{aligned} 0 \;=\; & S_2 = p\frac{(-x''(t)\,y'(t) + x'(t)\,y''(t))}{(x'(t))^2 + (y'(t))^2}\left(\frac{y'(t)}{x(t)} - \frac{x'(t)}{y(t)}\right) \\ & + \frac{1}{2}p(p-1)\left(\left(\frac{y'(t)}{x(t)}\right)^2 + \left(\frac{x'(t)}{y(t)}\right)^2\right) - p^2\frac{x'(t)\,y'(t)}{x(t)\,y(t)}. \end{aligned} \tag{1}$$

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In order to study the profile curves of such hypersurfaces we proceeded as in Alencar 1993, analyzing the trajectories of an associated vector field X. Each trajectory $\phi(t) = (u(t), v(t))$ of X is associated to a family M_λ of hypersurfaces generated by profile curves $\gamma_\lambda(t) = (\lambda x(t), \lambda y(t))$, determined by $\phi(t)$ up to homothety. The profile curves $\gamma(t)$ in the orbit space of these hypersufaces are one of the following types:

A) $\gamma(t)$ is one of the following half-straight line

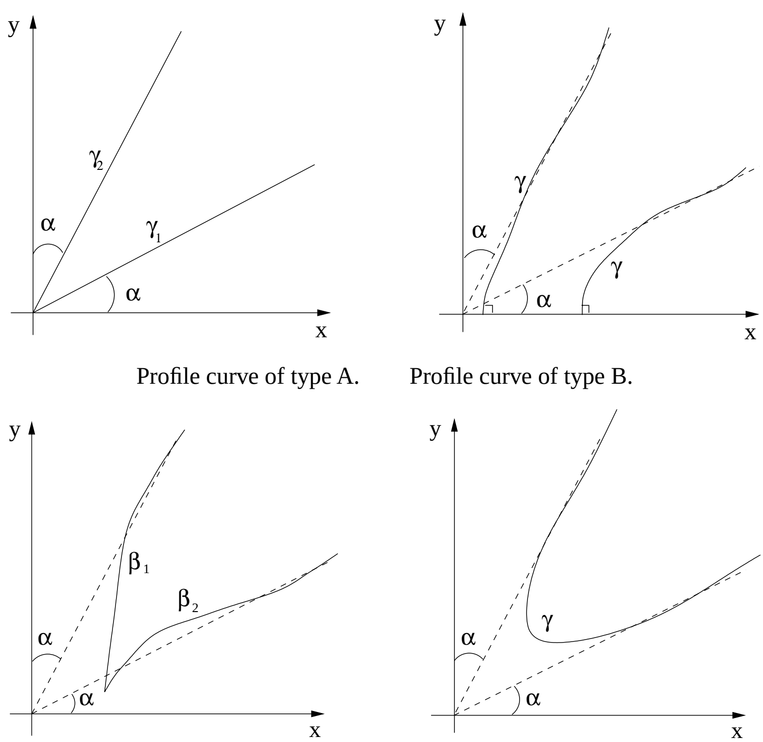
$$\gamma_1(t) = (\cos(\alpha)\, t, \sin(\alpha)\, t) \quad \text{or} \quad \gamma_2(t) = (\sin(\alpha)\, t, \cos(\alpha)\, t)$$

where $t \geq 0$ and $\alpha = \frac{1}{4} \arccos\left(\frac{3-2p}{2p-1}\right)$ (see figure 1);

B) $\gamma(t)$ is regular, intersects orthogonally one of the half-axes $x \geq 0$ or $y \geq 0$ and asymptotizes one of the half-straight lines in case A), when $t \longrightarrow +\infty$ or $t \longrightarrow -\infty$ (see figure 1);

C) $\gamma(t)$ is the union of two curves $\beta_1 : (-\infty, 0] \longrightarrow \Omega$ and $\beta_2 : [0, +\infty) \longrightarrow \Omega$, $\beta_1(0) = \beta_2(0)$ being a singularity. The curves β_i do not intesect the boundary of the orbit space, and asymptotizes the half-straight lines of the case A, when $t \longrightarrow \pm\infty$ (see figure 1);

D) $\gamma(t)$ is regular and does not intersect the boundary of the orbit space and asymptotizes both half-straight lines of the case A, when $t \longrightarrow \pm\infty$ (see figure 1)



Profile curve of type A. Profile curve of type B.

Profile curve of type C. Profile curve of type D.

Fig. 1 – Profile curves.

We will denote by C_α and $C_{\frac{\pi}{2}-\alpha}$ the cones generated by the half-straight lines of type A.

The main result of this work is the theorem below classifing $O(p+1) \times O(p+1)$-invariant hypersurfaces according to their profile curves.

CLASSIFICATION THEOREM. *The $O(p+1) \times O(p+1)$-invariant hypersurfaces in R^{2p+2} with $p > 1$ and zero scalar curvature belong to one of the following classes:*

1. *cones with a singularity in the origin of R^{2p+2}(type A).*
2. *hypersurfaces that have one orbit of singularities and that are asymptotic to both the cones C_α e $C_{\frac{\pi}{2}-\alpha}$ (type C).*
3. *regular hypersurfaces that are asymptotic to the cone C_α (type B).*
4. *regular hypersurfaces that are asymptotic to the cone $C_{\frac{\pi}{2}-\alpha}$ (type B).*
5. *regular hypersurfaces that are asymptotic to both cones C_α and $C_{\frac{\pi}{2}-\alpha}$ (type D).*

As a corollary we obtain the following result.

THEOREM A. *Let M^{2p+1} be an $O(p+1) \times O(p+1)$-invariant hypersurface in R^{2p+2}, complete and with zero scalar curvature. Then M is generated by a curve of type B or D. Moreover*

i) *If M is generated by a curve of type B, then M is embedded and asymptotic to one of the cones C_α or $C_{\frac{\pi}{2}-\alpha}$;*

ii) *If M is generated by a curve of type D, then M is embedded and asymptotic to both of the cones C_α and $C_{\frac{\pi}{2}-\alpha}$.*

The cones C_α and $C_{\frac{\pi}{2}-\alpha}$, generated by the half-straight lines in case A are characterized in the following theorem:

THEOREM B. *If M^{2p+1} is an $O(p+1) \times O(p+1)$-invariant hypersurface in R^{2p+2}, with zero scalar curvature whose profile curve makes a constant angle with the x-axes then M is one of the cones C_α or $C_{\frac{\pi}{2}-\alpha}$.*

This work is organized as follows. In section 2 we reduce the study of the profile curves $\gamma(t)$ of the invariant hypersurfaces in R^{2p+2}, with zero scalar curvature, to the study of the trajectory $\phi(t) = (u(t), v(t))$ of a vector field X. Then we use the qualitative theory of ordinary differential equations, together with a geometric analysis of the behavior of X, to obtain a description of its trajectories.

In section 3, we present sketches of the proofs of the theorems announced above.

2. ANALYSIS OF THE VECTOR FIELD X

The regular curves $(x(t), y(t))$ satisfing the equation $S_2 = 0$ are invariant by homotheties and, therefore, for each solution $\gamma(t)$ of (1) we have a family M_λ of invariant hypersurfaces with zero scalar curvature, generated by the curves $\gamma_\lambda(t) = (\lambda x(t), \lambda y(t))$. So we can apply the method developed in (Bombieri *et al.* 1969) to study the corresponding differential equation. Also note that, if a curve (x, y) is a solution of equation (1), then (y, x) is also a solution.

Without loss of generality, we may assume that the curves $\gamma(t)$ are parametrized by arc length. Therefore, when $y = y(x)$ we obtain

$$\frac{d^2y}{dx^2} = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)\left[-\frac{p(p-1))}{2}\left(\frac{y}{x}\left(\frac{dy}{dx}\right)^2+\frac{x}{y}\right)+p^2\frac{dy}{dx}\right]}{p\left(-x+y\frac{dy}{dx}\right)}. \tag{2}$$

Proceeding as in Bombieri *et al.* 1969 we introduce the parameters

$$u = \arctan\left(\frac{y}{x}\right) \text{ and } v = \arctan\left(\frac{y'}{x'}\right) \tag{3}$$

which are invariant by the homothety $(x, y) \longmapsto \lambda(x, y)$. Assuming $u' \neq 0$, we rewrite equation (1) as the system

$$\begin{aligned}
\frac{du}{dt} &= X_1(u,v) = -\frac{1}{4}p\sin(2u)[\sin(2u)-\sin(2v)]\\
\frac{dv}{dt} &= X_2(u,v) = \frac{1}{8}p[2(p-1)-\cos(2u-2v)+(2p-1)\cos(2u+2v)].
\end{aligned}$$

We associate to this system the vector field $X(u, v) = (X_1(u, v), X_2(u, v))$ in the (u, v)-plane.

Since our orbit space is the region Ω, we need information just for $x, y \geq 0$, corresponding to the region $R = \{(u, v)\ ; 0 \leq u \leq \frac{\pi}{2}, -\pi \leq v \leq \pi\}$ in the (u, v)-plane .We observe that X is bounded, π-periodic in both variables and invariant by a translation of $(\frac{\pi}{2}, \frac{\pi}{2})$. So, it is enough to analyse it in the interval $[0, \frac{\pi}{2}] \times [0, \pi]$.

In order to characterize the phase portrait of the field X we make a geometric study of its behaviour. This study gives us information about the increasing and decreasing intervals of the coordinates $u(t)$ and $v(t)$ of an orbit $\phi(t) = (u(t), v(t))$, the types of singularities that X presents and a transversality of X on special curves. This tranversality supplies barriers for the possible behaviors of those orbits of X.

These informations, together with the tubular flow theorem and Poincaré-Bendixson's theorem allow us to prove the following proposition, where we use the notation:

$$R_1 = \{u < v < \frac{\pi}{2} - u\} \cap \{0 < u < \frac{\pi}{4}\},$$

$$R_2 = \{0 \leq v < u\} \cap \{0 \leq v < \frac{\pi}{2} - u, \},$$

$$R_3 = \{\frac{\pi}{2} - u < v < u\} \cap \{\frac{\pi}{4} < u < \frac{\pi}{2}\},$$
$$R_4 = \{\frac{\pi}{2} - u < v \le \frac{\pi}{2}\} \cap \{u < v \le \frac{\pi}{2}\}$$

and

$$R_i^{-\pi} = R_i + (-\pi, 0)) \quad i = 1, ..., 4.$$

PROPOSITION 1. *The trajectories $\phi(t)$ of $X = (X_1, X_2)$ are defined for all values of t. In the region $R = \left\{(u, v) \in R^2; 0 \le u \le \frac{\pi}{2}, -\pi \le v \le \pi\right\}$ their possible behaviors is one of the following:*

1) *$\phi(t)$ is a vertical trajectory with α-limit $\left(0, -\frac{\pi}{2}\right)$ and ω-limit $\left(0, \frac{\pi}{2}\right)$, or a vertical trajectory with α-limit $\left(\frac{\pi}{2}, 0\right)$ and ω-limit $\left(\frac{\pi}{2}, \pi\right)$,or still a vertical trajectory with α-limit $\left(\frac{\pi}{2}, -\pi\right)$ and ω-limit $\left(\frac{\pi}{2}, 0\right)$.*

2) *$\phi(t)$ is a vertical half-trajectory with α-limit $\left(0, -\frac{\pi}{2}\right)$, or a vertical half-trajectory with ω-limit $\left(0, \frac{\pi}{2}\right)$.*

3) *$\phi(t)$ is a trajectory in $\left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right)$ with α-limit $\left(\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha\right)$ and ω-limit (α, α) going through the points of $J_1 = \left\{\left(u, \frac{\pi}{2} - u\right); 0 < u < \frac{\pi}{2}\right\}$ where $\alpha = \frac{1}{4} \arccos\left(\frac{3-2p}{2p-1}\right)$.*

4) *$\phi(t)$ is a connection of saddle points contained in the region $R_3 \cup R_4$ with α-limit $\left(\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha\right)$ and ω-limit $\left(0, \frac{\pi}{2}\right)$.*

5) *$\phi(t)$ is a connection of saddle points contained in the region $R_1 \cup R_2$ with α-limit $\left(0, \frac{\pi}{2}\right)$ and ω-limit (α, α).*

6) *$\phi(t)$ is a connection of saddle points contained in the region $R_1 \cup R_2$ with α-limit $\left(\frac{\pi}{2}, 0\right)$ and ω-limit (α, α).*

7) *$\phi(t)$ is a connection of saddle points contained in the region $R_3 \cup R_4$ with α-limit $\left(\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha\right)$ and ω-limit $\left(\frac{\pi}{2}, 0\right)$.*

8) *$\phi(t)$ is a trajectory contained in the region $R_1 \cup R_2 \cup (0, \frac{\pi}{2}) \times [-\frac{\pi}{2}, 0] \cup R_4^{-\pi} \cup R_3^{-\pi}$, with α-limit $\left(-\frac{\pi}{2} - \alpha, -\frac{\pi}{2} - \alpha\right)$ and ω-limit (α, α).*

9) *$\phi(t)$ is an orbit, or part of one, obtained by a translation of $(0, \pm\pi)$, of one of the orbts given in the itens 1-8 .*

3. $\mathbf{O(p+1) \times O(p+1)}$-INVARIANT HYPERSURFACES IN $\mathbf{R}^{2p+2}$

The hypersurfaces of type A (item 1 of the Classification theorem) are given by the cones C_α e $C_{\frac{\pi}{2}-\alpha}$ and characterized in Theorem B, whose proof consists in to use that, if $\gamma(t) = (x(t), y(t))$ is a solution with $y(t) = \tan\alpha x(t)$, then it satisfies the equation $0 = 1 + \left(\frac{-4p+2}{p-1}\right)\frac{\sin^2 2\alpha}{4}$. This, together with the fact that γ is parametrized by arc length, give us the result.

Theorem A follows from the Classification theorem, Lemma 1 and Remark 1 below.

LEMMA 1. *Let $\phi(t) = (u(t), v(t))$ be a trajectory with α-limit $(\frac{\pi}{2} - \alpha, -\frac{\pi}{2} - \alpha)$ and ω-limit (α, α). Let $\gamma(t) = (x(t), y(t))$ be the associated profile curve. Then $\phi(t)$ intersects the segment $l = \left\{\left(\frac{\pi}{4}, v\right) : -\pi < v < \frac{\pi}{2}\right\}$ exactly once, so $\gamma(t)$ intersects the diagonal $y = x$ exactly once. Therefore, γ does not possess self-intersections and the hypersurface generated by γ is embedded and complete.*

REMARK 1. *If γ is a profile curve associated to a connection of saddle points, then γ is a graph over one of the axes x or y, and intersects it orthoganally. Therefore the hypersurface generated by γ is embedded and complete.*

The proof of the Classification theorem is a consequence of the Proposition 1, together with the remark below:

REMARK 2. *For $0 < v < \frac{\pi}{2}$ we have $x'(t) \neq 0,\ y'(t) \neq 0$ and so we can see the profile curve as a graph (or union of graphs when $\gamma(t) = (x(t), y(t))$ has singularities) of a function $y = y(x)$ or $x = x(y)$. We will assume without loss of generality, that $y = y(x)$. In this case, equation (2) tells us that there are singularities at the zeros of the equation*

$$x - y\frac{dy}{dx} = 0.$$

They correspond to the coordinates (u, v) with $v = \frac{\pi}{2} - u$.

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