On the geometry of Poincaré's problem for one-dimensional projective foliations

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ABSTRACT

We consider the question of relating extrinsic geometric characters of a smooth irreducible complex projective variety, which is invariant by a one-dimensional holomorphic foliation on a complex projective space, to geometric objects associated to the foliation.

Key words: holomorphic foliations, invariant varieties, polar classes, degrees.

1 INTRODUCTION

H. Poincaré treated, in (1891), the question of bounding the degree of an algebraic curve, which is a solution of a foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ with rational first integral, in terms of the degree of the foliation. This problem has been considered more recently in the following formulation: to bound the degree of an irreducible algebraic curve S, invariant by a foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$, in terms of the degree of the foliation.

Simple examples show that, when S is a dicritical separatrix of $\mathcal{F}$, the search for a positive solution to the problem is meaningless. The obstruction in this case was given by M. Brunella in (1997), and reads: the number $\int_S c_1(N_{\mathcal{F}}) - S\cdot S$ may be negative if S is a dicritical separatrix (here, $N_{\mathcal{F}}$ is the normal bundle of the foliation). More than that, A. Lins Neto constructs, in (2000), some remarkable families of foliations on $\mathbb{P}^2_{\mathbb{C}}$ providing counterexamples for this problem, all involving singular separatrices and dicritical singularities.

However, as was shown in (Brunella 1997), when S is a non-dicritical separatrix, the number $\int_S c_1(N_{\mathcal{F}}) - S\cdot S$ is nonnegative and, in $\mathbb{P}^2_{\mathbb{C}}$, this means $d^0(\mathcal{F})+2 \geq d^0(S)$, where $d^0(\mathcal{F})$ and $d^0(S)$ are the degrees of the foliation and of the curve, respectively. Another solution to the problem, in the non-dicritical case, was given by M.M. Carnicer in (1994), using resolution of singularities.

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Let us now consider one-dimensional holomorphic foliations on $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 2$, that is, morphisms $\mathcal{F} : \mathcal{O}(m) \longrightarrow \mathrm{T}\mathbb{P}^n_{\mathbb{C}}$, $m \in \mathbb{Z}$, $m \leq 1$, with singular set of codimension at least 2. We write $m = 1 - d^0(\mathcal{F})$ and call $d^0(\mathcal{F}) \geq 0$ the degree of $\mathcal{F}$. From now on we will consider $d^0(\mathcal{F}) \geq 2$. This is the characteristic number associated to the foliation.

On the other hand, if we consider $\mathcal{F}$-invariant algebraic varieties $\mathbf{V} \xrightarrow{\mathbf{i}} \mathbb{P}^n_{\mathbb{C}}$, it is natural to consider other characters associated to $\mathbf{V}$, not just its degree. This is the point of view we address. More precisely, we pose the question of relating extrinsic geometric characters of $\mathbf{V}$ to geometric objects associated to $\mathcal{F}$.

This approach produces some interesting results. Let us illustrate the two-dimensional situation. Suppose we have an $\mathcal{F}$-invariant irreducible plane curve S. We associate to $\mathcal{F}$ a tangency divisor $\mathcal{D}_{\mathcal{H}}$ (depending on a pencil $\mathcal{H}$), which is a curve of degree $d^0(\mathcal{F}) + 1$ and contains the first polar locus of S. Computing degrees we arrive at $d^0(S) \leq d^0(\mathcal{F}) + 2$ in case S is smooth, and at $d^0(S)(d^0(S) - 1) - \sum_{p \in sing(S)}(\mu_p - 1) \leq (d^0(\mathcal{F}) + 1)d^0(S)$ in case S is singular, where μ_p is the Milnor number of S at p. This allows us to recover a result of D. Cerveau and A. Lins Neto (1991), which states that if S has only nodes as singularities, then $d^0(S) \leq d^0(\mathcal{F}) + 2$, regardless of the singularities of $\mathcal{F}$ being dicritical or non-dicritical.

In the higher dimensional situation, we obtain relations among polar classes of $\mathcal{F}$-invariant smooth varieties and the degree of the foliation.

2 THE TANGENCY DIVISOR OF $\mathcal{F}$ WITH RESPECT TO A PENCIL

Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $\mathbb{P}^n_{\mathbb{C}}$ of degree $d^0(\mathcal{F}) \geq 2$, with singular set of codimension at least 2. We associate a *tangency divisor* to $\mathcal{F}$ as follows:

Choose affine coordinates $(z_1, \ldots, z_n)$ such that the hyperplane at infinity, with respect to these, is not $\mathcal{F}$-invariant, and let $X = gR + \sum_{i=1}^{n} Y_i \frac{\partial}{\partial z_i}$ be a vector field representing $\mathcal{F}$, where $R = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}$, $g(z_1, \ldots, z_n) \not\equiv 0$ is homogeneous of degree $d^0(\mathcal{F})$ and $Y_i(z_1, \ldots, z_n)$ is a polynomial of degree $\leq d^0(\mathcal{F})$, $1 \leq i \leq n$. Let H be a generic hyperplane in $\mathbb{P}^n_{\mathbb{C}}$. Then, the set of points in H which are either singular points of $\mathcal{F}$ or at which the leaves of $\mathcal{F}$ are not transversal to H is an algebraic set, noted $tang(H, \mathcal{F})$, of dimension $n - 2$ and degree $d^0(\mathcal{F})$ (observe that $g(z_1, \ldots, z_n) = 0$ is precisely $tang(H_\infty, \mathcal{F})$).

Definition. *Consider a pencil of hyperplanes $\mathcal{H} = \{H_t\}_{t \in \mathbb{P}^1_{\mathbb{C}}}$, with axis L^{n-2}. The tangency divisor of $\mathcal{F}$ with respect to $\mathcal{H}$ is*

$$\mathcal{D}_{\mathcal{H}} = \bigcup_{t \in \mathbb{P}^1_{\mathbb{C}}} tang(H_t, \mathcal{F}).$$

Lemma 2.1. *$\mathcal{D}_{\mathcal{H}}$ is a (possibly singular) hypersurface of degree $d^0(\mathcal{F}) + 1$.*

Proof. Let p be a point in L^{n-2}, the axis of the pencil. If $p \in sing(\mathcal{F})$ then p is necessarily in $\mathcal{D}_{\mathcal{H}}$, otherwise p is a regular point of $\mathcal{F}$. In this case, if $\mathcal{L}$ is the leaf of $\mathcal{F}$ through p, then either $\mathrm{T}_p\mathcal{L} \subset L^{n-2}$ or, $\mathrm{T}_p\mathcal{L}$ together with L^{n-2} determine a hyperplane $H_\alpha \in \mathcal{H}$, and hence we have

$p \in tang(H_\alpha, \mathcal{F}) \subset \mathcal{D}_{\mathcal{H}}$, so that $L^{n-2} \subset \mathcal{D}_{\mathcal{H}}$. Now, let $p \in L^{n-2}$ be a regular point of $\mathcal{F}$ and choose a generic line ℓ, transverse to L^{n-2}, passing through p and such that L^{n-2} and ℓ determine a hyperplane H_β, distinct from H_α. This line ℓ meets $\mathcal{D}_{\mathcal{H}}$ at p and at $d^0(\mathcal{F})$ further points, counting multiplicities, corresponding to the intersections of ℓ with $tang(H_\beta, \mathcal{F})$. Hence $\mathcal{D}_{\mathcal{H}}$ has degree $d^0(\mathcal{F})+1$. □

EXAMPLE. If we consider the two-dimensional Jouanolou's example

$$\dot{x} = y^{d^0(\mathcal{F})} - x^{d^0(\mathcal{F})+1}$$

$$\dot{y} = 1 - yx^{d^0(\mathcal{F})}$$

and the pencil $\mathcal{H} = \{(at, bt) : t \in \mathbb{C}\ ,\ (a:b) \in \mathbb{P}^1_{\mathbb{C}}\}$, a straightforward manipulation shows that $\mathcal{D}_{\mathcal{H}}$ is given, in homogeneous coordinates $(X:Y:Z)$ in $\mathbb{P}^2_{\mathbb{C}}$, by

$$Y^{d^0(\mathcal{F})+1} - XZ^{d^0(\mathcal{F})} = 0.$$

3 $\mathcal{F}$-INVARIANT SMOOTH IRREDUCIBLE VARIETIES

Let us recall some facts about polar varieties and classes (Fulton 1984). If $\mathbf{V} \xrightarrow{\mathbf{i}} \mathbb{P}^n_{\mathbb{C}}$ is a smooth irreducible algebraic subvariety of $\mathbb{P}^n_{\mathbb{C}}$, of dimension $n-k$, and L^{k+j-2} is a linear subspace, then the j-th polar locus of $\mathbf{V}$ is defined by

$$\mathcal{P}_j(\mathbf{V}) = \left\{q \in \mathbf{V} |\ \dim\left(\mathrm{T}_q\mathbf{V} \cap L^{k+j-2}\right) \geq j-1\right\}$$

for $0 \leq j \leq n-k$. If L^{k+j-2} is a generic subspace, the codimension of $\mathcal{P}_j(\mathbf{V})$ in $\mathbf{V}$ is precisely j. The j-th class, $\varrho_j(\mathbf{V})$, of $\mathbf{V}$ is the degree of $\mathcal{P}_j(\mathbf{V})$ and, since the cycle associated to $\mathcal{P}_j(\mathbf{V})$ is

$$\left[\mathcal{P}_j(\mathbf{V})\right] = \sum_{i=0}^{j}(-1)^i \binom{n-k-i+1}{j-i} c_i(\mathbf{V})c_1(\mathbf{i}^*\mathcal{O}(1))^{j-i}$$

we have

$$\varrho_j(\mathbf{V}) = \int_{\mathbf{V}} \sum_{i=0}^{j} (-1)^i \binom{n-k-i+1}{j-i} c_i(\mathbf{V})c_1(\mathbf{i}^*\mathcal{O}(1))^{n-k-i} \quad , \quad 0 \leq j \leq n-k.$$

LEMMA 3.1. *Let* $\mathbf{V}$ *be a smooth irreducible algebraic variety of dimension* $n-k$*,* $\mathcal{F}$*-invariant and not contained in* $sing(\mathcal{F})$*. Then*

$$\mathcal{P}_{n-k}(\mathbf{V}) \subset \mathcal{D}_{\mathcal{H}} \quad \text{and} \quad \mathcal{P}_0(\mathbf{V}) = \mathbf{V} \not\subset \mathcal{D}_{\mathcal{H}}.$$

PROOF. Let us first assume $\mathbf{V}$ is a linear subspace of $\mathbb{P}^n_{\mathbb{C}}$. In this case $\mathcal{P}_j = \emptyset$, for $j \geq 1$, so the first assertion of the lemma is meaningless. Assume then $\mathbf{V}$ is not a linear subspace and choose a pencil of hyperplanes $\mathcal{H} = \{H_t\}_{t\in\mathbb{P}^1_{\mathbb{C}}}$, with axis L^{n-2} generic, so that $\mathrm{codim}(\mathcal{P}_{n-k}(\mathbf{V}), \mathbf{V}) = n-k$. If $q \in \mathcal{P}_{n-k}(\mathbf{V})$, then $\mathrm{T}_q\mathbf{V}$ meets L^{n-2} in a subspace W of dimension at least $n-k-1$. If $\mathrm{T}_q\mathbf{V} \subset L^{n-2}$ then any hyperplane $H_t \in \mathcal{H}$ contains $\mathrm{T}_q\mathbf{V}$, if not, a line $\ell \subset \mathrm{T}_q\mathbf{V}$, $\ell \not\subset L^{n-2}$, $\ell \cap W$ consisting of a point determines, together with L^{n-2}, a hyperplane $H_t \in \mathcal{H}$ such that $\mathrm{T}_q\mathbf{V} \subset H_t$. Since $\mathbf{V}$ is $\mathcal{F}$-invariant, we have $\mathrm{T}_q\mathcal{L} \subset \mathrm{T}_q\mathbf{V} \subset H_t$, in case q is not a singular point of $\mathcal{F}$, where $\mathcal{L}$ is the leaf of $\mathcal{F}$ through q. This implies $q \in tang(H_t, \mathcal{F}) \subset \mathcal{D}_{\mathcal{H}}$, so that $\mathcal{P}_{n-k}(\mathbf{V}) \subset \mathcal{D}_{\mathcal{H}}$. Also, it follows from the definition of $\mathcal{D}_{\mathcal{H}}$ that $\mathbf{V}$ is not contained in it. □

THEOREM I. *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $\mathbb{P}^n_{\mathbb{C}}$ of degree $d^0(\mathcal{F}) \geq 2$, with singular set of codimension at least 2, and let $\mathbf{V}$ be an $\mathcal{F}$-invariant smooth irreducible algebraic variety, of dimension $n-k$, which is not a linear subspace of $\mathbb{P}^n_{\mathbb{C}}$, and not contained in $sing(\mathcal{F})$. Suppose $\mathcal{P}_{n-k-j}(\mathbf{V}) \subset \mathcal{D}_{\mathcal{H}}$ but $\mathcal{P}_{n-k-j-1}(\mathbf{V}) \not\subset \mathcal{D}_{\mathcal{H}}$, for some $0 \leq j \leq n-k-1$. Then*

$$\frac{\varrho_{n-k-j}(\mathbf{V})}{\varrho_{n-k-j-1}(\mathbf{V})} \leq d^0(\mathcal{F}) + 1.$$

PROOF. Observe that we may assume $\mathcal{P}_{n-k-j}(\mathbf{V}) \subset \mathcal{P}_{n-k-j-1}(\mathbf{V})$ and hence

$$\mathcal{P}_{n-k-j}(\mathbf{V}) \subseteq \mathcal{D}_{\mathcal{H}} \cap \mathcal{P}_{n-k-j-1}(\mathbf{V})$$

Bézout's Theorem then gives

$$\varrho_{n-k-j}(\mathbf{V}) \leq (d^0(\mathcal{F}) + 1)\varrho_{n-k-j-1}(\mathbf{V}).$$ □

COROLLARY 1. *Let $\mathbf{V}^{n-k}_{(d_1,\ldots,d_k)} \not\subset sing(\mathcal{F})$ be a smooth irreducible complete intersection in $\mathbb{P}^n_{\mathbb{C}}$, which is not a linear subspace, defined by $F_1 = 0, \ldots, F_k = 0$ where $F_\ell \in \mathbb{C}[z_0, \ldots, z_n]$ is homogeneous of degree d_ℓ, $1 \leq \ell \leq k$ and $\mathcal{F}$-invariant, where $\mathcal{F}$ is as in Theorem I. If $\mathcal{P}_{n-k-j}(\mathbf{V}^{n-k}_{(d_1,\ldots,d_k)}) \subset \mathcal{D}_{\mathcal{H}}$ but $\mathcal{P}_{n-k-j-1}(\mathbf{V}^{n-k}_{(d_1,\ldots,d_k)}) \not\subset \mathcal{D}_{\mathcal{H}}$ then*

$$d^0(\mathcal{F}) + 1 \geq \frac{\mathcal{W}^{(k)}_{n-k-j}(d_1 - 1, \ldots, d_k - 1)}{\mathcal{W}^{(k)}_{n-k-j-1}(d_1 - 1, \ldots, d_k - 1)}$$

where $\mathcal{W}^{(k)}_\delta$ is the Wronski (or complete symmetric) function of degree δ in k variables

$$\mathcal{W}^{(k)}_\delta(X_1, \ldots, X_k) = \sum_{i_1+\cdots+i_k=\delta} X_1^{i_1} \ldots X_k^{i_k}.$$

PROOF. Immediate since $\varrho_i(\mathbf{V}^{n-k}_{(d_1,\ldots,d_k)}) = (d_1.\cdots.d_k)\mathcal{W}^{(k)}_i(d_1 - 1, \ldots, d_k - 1)$. □

Observe that if $\mathbf{V}$ is a smooth irreducible hypersurface, this reads $d^0(\mathcal{F}) + 2 \geq d^0(\mathbf{V})$. In (Soares 1997) we showed $d^0(\mathcal{F}) + 1 \geq d^0(\mathbf{V})$, but assumed $\mathcal{F}$ to be a non-degenerate foliation on $\mathbb{P}^n_{\mathbb{C}}$.

Also, in (Soares 2000) the following estimate is obtained, provided $n-k$ is odd and $\mathbf{i}^*\mathcal{F}$ is non-degenerate: if $1 \leq k \leq n-2$ then

$$d^0(\mathcal{F}) \geq \frac{\varrho_{n-k}(\mathbf{V}^{n-k}_{(d_1,\ldots,d_k)})}{\varrho_{n-k-1}(\mathbf{V}^{n-k}_{(d_1,\ldots,d_k)})}$$

We remark that this estimate is sharper than that given in Corollary 1.

4 THE TWO-DIMENSIONAL CASE

As pointed out in Corollary 1, whenever we have a smooth irreducible $\mathcal{F}$-invariant plane curve S, the relation $d^0(S) \leq d^0(\mathcal{F})+2$ holds because $\varrho_1(S) = d^0(S)(d^0(S)-1)$, regardless of the nature of the singularities of $\mathcal{F}$, provided $sing(\mathcal{F})$ has codimension two.

In order to treat the case of arbitrary irreducible $\mathcal{F}$-invariant curves, let us recall the definition (see R. Piene 1978) of the *class* of a (possibly singular) irreducible curve S in $\mathbb{P}^2_{\mathbb{C}}$. We let S_{reg} denote the regular part of S and, for a generic point p in $\mathbb{P}^2_{\mathbb{C}}$, we consider the subset $\mathcal{Q}$ of S_{reg} consisting of the points q such that $p \in T_q S_{reg}$. The closure $\mathcal{P}_1$ of $\mathcal{Q}$ in S is the first polar locus of S, and the *class* $\varrho_1(S)$ of S is its degree. $\mathcal{P}_1$ is a subvariety of codimension 1 whose degree is given by Teissier's formula (Teissier 1973):

$$\varrho_1(S) = d^0(S)(d^0(S)-1) - \sum_q (\mu_q + m_q - 1)$$

where the summation is over all singular points q of S, μ_q denotes the Milnor number of S at q and m_q denotes the multiplicity of S at q. Because $\mathcal{P}_1$ is a finite set of regular points in S, revisiting Lemma 3.1 we conclude:

$$\mathcal{P}_1 \subseteq \mathcal{D}_{\mathcal{H}} \cap S.$$

Also, $sing(S) \subseteq sing(\mathcal{F})$, so that

$$sing(S) \subseteq \mathcal{D}_{\mathcal{H}} \cap S$$

and hence

$$\mathcal{P}_1 \cup sing(S) \subseteq \mathcal{D}_{\mathcal{H}} \cap S.$$

It follows from Bézout's theorem that

$$\varrho_1(S) + \sum_q m_q \leq (d^0(\mathcal{F})+1)d^0(S)$$

Therefore we obtain the

THEOREM II. *Let S be an irreducible curve, of degree $d^0(S) > 1$, invariant by a foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$, of degree $d^0(\mathcal{F}) \geq 2$ with $sing(\mathcal{F})$ of codimension 2. Then*

$$d^0(S)(d^0(S)-1) - \sum_q (\mu_q - 1) \leq (d^0(\mathcal{F})+1)d^0(S)$$

where the summation extends over all singular points q of S. □

This gives at once the following result, first obtained by Cerveau and Lins Neto (1991);

COROLLARY 2. *If all the singularities of S are ordinary double points (so that $\mu_q = 1$) then*

$$d^0(S) \leq d^0(\mathcal{F}) + 2. \qquad \square$$

Theorem II illustrates one obstruction to solving Poincaré's problem in general, since we cannot estimate the sum $\sum_q (\mu_q - 1)$ when dicritical singularities are present. However, if S is an irreducible $\mathcal{F}$-invariant algebraic curve, which is a non-dicritical separatrix, then it follows from (Brunella 1997) that

$$\sum_q (\mu_q - 1) \leq \sum_q \sum_{i=1}^{r_q} GSV(\mathcal{F}, B_i^q, q) - \sum_q r_q$$

where the sum is over all singular points q of S, $B_1^q, \ldots, B_{r_q}^q$ are the analytic branches of S at q, and GSV denotes the Gomez-Mont/Seade/Verjovsky index.

REMARK. Let S be a non-dicritical separatrix of $\mathcal{F}$, so that $d^0(S) \leq d^0(\mathcal{F}) + 2$. Assume equality holds in the expression in Theorem II, which amounts to

$$d^0(S)(d^0(S) - d^0(\mathcal{F}) - 2) = \sum_q (\mu_q - 1) \geq 0.$$

Hence we conclude $d^0(S) = d^0(\mathcal{F}) + 2$ and S has only ordinary double points as singularities. $\square$

5 $\mathcal{F}$-INVARIANT SMOOTH IRREDUCIBLE CURVES

We have the following immediate consequence of Corollary 1: if we consider an $\mathcal{F}$-invariant smooth one-dimensional complete intersection $S = \mathbf{V}^{n-(n-1)}_{(d_1,\ldots,d_{(n-1)})} \not\subset sing(\mathcal{F})$, then

$$d_1 + \cdots + d_{n-1} \leq d^0(\mathcal{F}) + n$$

so that

$$d^0(S) \leq \left(\frac{d^0(\mathcal{F}) + n}{n-1}\right)^{n-1}$$

provided codim $sing(\mathcal{F}) \geq 2$. In the general case we have:

COROLLARY 3. *Let $S \not\subseteq sing(\mathcal{F})$ be an $\mathcal{F}$-invariant smooth irreducible curve of degree $d^0(S) > 1$, where $\mathcal{F}$ is a one-dimensional holomorphic foliation on $\mathbb{P}^n_{\mathbb{C}}$ of degree $d^0(\mathcal{F}) \geq 2$, with singular set of codimension at least 2. Then the first class $\varrho_1(S)$ of S satisfies*

$$\varrho_1(S) \leq (d^0(\mathcal{F}) + 1)d^0(S),$$

the geometric genus g of S satisfies

$$g \leq \frac{(d^0(\mathcal{F}) - 1)d^0(S)}{2} + 1.$$

Also, if $N(\mathcal{F}, S)$ is the number of singularities of $\mathcal{F}$ along S, then

$$N(\mathcal{F}, S) \leq (d^0(\mathcal{F}) + 1)d^0(S).$$

PROOF. Since S is a curve which is not a line, we have to consider only $\varrho_0(S) = d^0(S)$ and $\varrho_1(S)$. The first inequality follows immediately from Theorem I. To bound the genus we observe that Lefschetz' theorem on hyperplane sections (Lamotke 1981) gives

$$\varrho_1(S) = 2d^0(S) + 2g - 2$$

and the second inequality follows. On the other hand, since S is irreducible and not contained in $sing(\mathcal{F})$, Whitney's finiteness theorem for algebraic sets (Milnor 1968) implies that $S \setminus sing(\mathcal{F})$ is connected, and hence $N(\mathcal{F}, S)$ is necessarily finite. Also,

$$sing(\mathcal{F}) \cap S \subset \mathcal{D}_{\mathcal{H}} \cap S$$

and Bézout's theorem implies

$$N(\mathcal{F}, S) \leq (d^0(\mathcal{F}) + 1)d^0(S). \qquad \square$$

The first class of a smooth irreducible curve S in $\mathbb{P}^n_{\mathbb{C}}$ was calculated by R. Piene (1976), and is as follows:

$$\varrho_1(S) = 2(d^0(S) + g - 1) - \kappa_0$$

where g is the genus of S and $\kappa_0 \geq 0$ is an integer, called the $0 - th$ stationary index. It follows from Theorem I that:

COROLLARY 4. *With the same hypothesis of Corollary 3*

$$2d^0(S) - \chi(S) - \kappa_0 \leq (d^0(\mathcal{F}) + 1)d^0(S). \qquad \square$$

REMARK ON EXTREMAL CURVES. We can obtain an estimate for $d^0(S)$ in terms of $d^0(\mathcal{F})$ and $n \geq 3$, provided S is non-degenerate (that is, is not contained in a hyperplane) and *extremal* (that is, the genus of S attains Castelnuovo's bound). Recall that, for S a smooth non-degenerate curve in $\mathbb{P}^n_{\mathbb{C}}$ of degree $d^0(S) \geq 2n$, Castenuovo's bound is (Arbarello et al. 1985):

$$g \leq \frac{m(m-1)}{2}(n-1) + m\epsilon,$$

where

$$d^0(S) - 1 = m(n-1) + \epsilon.$$

The inequality

$$g \leq \frac{(d^0(\mathcal{F}) - 1)d^0(S)}{2} + 1$$

together with S extremal give, performing a straightforward manipulation:

$$d^0(S) \leq 2(d^0(\mathcal{F}) - 1)(n-1) + \frac{(n-1)(n+2)}{n}. \qquad \square$$

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RESUMO

Consideramos o problema de relacionar carateres geométricos extrínsecos de uma variedade projetiva lisa e irredutível, que é invariante por uma folheação holomorfa de dimensão um de um espaço projetivo complexo, a objetos geométricos associados à folheação.
Palavras-chave: folheações holomorfas, variedades invariantes, classes polares, graus.

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